

## WHAT DOES THIS PROGRAM DO, ANYWAY?

### 1. FIRST, A LITTLE ABOUT LENS FUNCTIONS.

Suppose you wanted to shape a parallel wave front so that it converged to a point at a distance  $f$  away from some phase changing device, like a thin lens or phase only optical device. (The latter type of device can rarely changes phase by more than  $2\pi$ , but with coherent laser light thats all you need.)

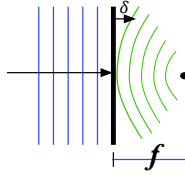


FIGURE 1. Phase at radius  $r$  is advanced by  $\delta$ .

Since the radius of the arc is  $f$  we know that  $\delta = f - \sqrt{f^2 - r^2}$  and if  $f \gg r$  then we can write  $\delta \approx r^2/2f$ . So one will get away with writing the following quadratic "chirp" and calling it a lens function.

$$(1) \quad e^{-i\frac{2\pi\delta}{\lambda}} = e^{-i\frac{\pi r^2}{\lambda f}}$$

The minus sign in front of the  $i$  is arbitrary and defined to be consistent with the usage above. The phase is of course actually delayed, so it is just delayed less than the center. A thin lens does this by just being thicker in the center. A phase only light valve has an image electronically sent to it that looks generally like the following. The map below is actually a representation of  $(2\pi r^2/\lambda f)_{\text{modulo}(2\pi)}$ .

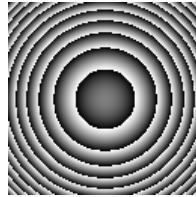


FIGURE 2. what is sent to light valve and interpreted by it as phase shift.

The image above is transmitted to a device (light valve) which produces a by the same process (or very similar) to the process that produces the image on the face of your smart phone. The screen may be reflective or transmissive but effectively produces a representation of the image sent to the screen. The screen may present an image of varying intensity of light (like your phone) or in special cases just change the phase of the transmitted light. The phase turns out to contain more information than the amplitude, and is more interesting. It is actually quite difficult to modulate both the phase and amplitude of the beam directly. However, there are ways. They are not covered in this monograph.

The lens function described above needs to be created, stored, and interpreted within your computer before it can be put on your display device. The display device and the internal representation of your data are both rasterized to pixels, and we assume that the pitch of both is the same. We will, in this document, call the pitch  $\Delta$  and the (0,0) pixel is in the center of the screen. The screen representation is obviously not so small but you must pretend that it is representing something very small. Positive is to the right and up.

Equation (1) above is clearly describing a field of complex numbers. And so the internal representation of the field is as complex numbers. Every pixel is represented by a pair of double precision numbers: the real part and the imaginary part. These numbers are called the "Big Array" for good reason. In general these numbers are interpreted and presented to the screen buffer for your enjoyment, based upon the selection made in the "view options" panel. When you select the real part for display, only the real part of the number field is presented to the screen buffer. Selecting "magnitude", "masquerade" or "phase" does not change the stored numbers, but only the content of the screen buffer.

There is a menu selection which will produce a lens function for you when you provide appropriate numbers to the window. We should look at how this is done to get a grasp of the program.

The  $N$  by  $N$  array is numbered  $-N/2 \leq x \leq N/2 - 1$  and  $-N/2 - 1 \leq y \leq N/2$ . The extra rows are of the left and top. A number  $s = 2\pi\Delta^2/\lambda f$  is calculated from the window fields and  $\cos(s(x^2 + y^2))$  is put into the real part and  $\sin(s(x^2 + y^2))$  into the imaginary part. *Well actually, it's calculated in stages into a temporary array and then just put into the big array, or if you wish it can be multiplied into the current contents of the big array.*

### 1.1. The Fourier transform is central to understanding coherent optics optics.

In figure (3) we portray coherent light passing through a transmissive medium on the left. If we focus our attention on the point  $u$  on the focal plane of the lens, we realize that all of the light along the blue line will arrive at the point  $u$  simultaneously. But the light at the lower end of the blue line clearly left the medium on the left before that further up on the line. So the amplitude of the light at the point  $u$  at some instant is the integral of the light which left from the bottom of the medium earlier than that which left from the top. As measured from the center ( $x = 0$ ) we would expect the relative lag ( $\delta$ ) of the wave front to be  $\delta = x \sin(\theta)$ . Where  $\theta$  is the angle between the horizontal center line and the red line. It is also the angle between the vertical  $x$  axis and the blue wave front

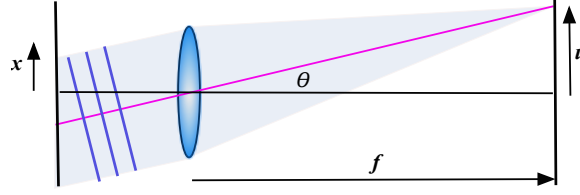


FIGURE 3. all the light along the blue line arrives at the point  $u$  at the same time.

lines. The phase shift due to this lag is  $2\pi\delta/\lambda = 2\pi x \sin(\theta)/\lambda$ . If we make the small angle approximation that  $\theta \approx \sin(\theta) \approx \tan(\theta) = u/f$  which translates to the complex numbers  $\exp[i2\pi xu/f\lambda]$ . So the electric field intensity at the point  $u$  would be proportional to the following integral.

$$\int f(x) e^{i\frac{2\pi}{f\lambda}xu} dx$$

This looks a lot like a Fourier transform and would be proportional to one if the phase at the focal plane was flat. It is instead quadratically dependent upon the distance from the center point.

We can flatten the phase to produce a real Fourier transform by either positioning the  $x$  plane so that it too is a distance  $f$  from the lens, or alternatively moving it to the lens and putting a duplicate lens at the  $u$  plane. The latter option produces an shorter path of the optical bench as well as fitting into the following discussion better.



FIGURE 4. Two lens short Fourier system.

**1.2. the trick.** So let's suppose you have a simple image like a letter of the alphabet visible on the window of this program. If you select the "Inv. Fourier" button the image changes accordingly. If this image were to be presented in its entirety and the lens system shown in figure (4) used then that letter would appear at plane two. In this case the second lens is not necessary since we are looking directly (via a charged coupled device or paper screen) at the image and losing the phase information. One may wonder if the physical lens at plane one is absolutely necessary since the lens function can be superimposed upon the Fourier transform. You can replace it. In fact the final lens can be eliminated as well by putting an appropriate lens function upon the letter before performing the Fourier transform. [ta da] A hologram.

Well actually its hard to put all of the phase and amplitude information on a light valve. If just the phase information is put on the valve the resulting view is a very edge enhanced image.

We can use the program to simulate these effects. The "phase only" option makes all values in the big array have magnitude one. Almost certainly increasing the magnitude of the higher spacial frequencies by a huge factor. If you how select the "direct propagation" option you see what would result from the light traveling the selected distance [step]. What follows is a discussion of the theory behind the transmission of light. You should not be surprised to see that the Fast Fourier Transform is in the center of much of the discussion.

## 2. PROPAGATION OF THE LIGHT

Lets start by looking at how to derive the Fresnel equation. By doing so we derive some very useful relations along the way. We want to start with a plane at  $z = 0$  with coordinates  $(u, v)$  on that plane. The amplitude of the light on that plane is  $A_0(u, v)$ . We can expect that the function at some point  $(x, y, z)$  following this in plane is described by a sum of plane waves having the form  $\exp[i(xk_x + yk_y + zk_z)]$  and having wave vector components  $(k_x, k_y, k_z)$ . We have illuminated the initial plane with coherent laser light having wave number  $k = 2\pi/\lambda = \sqrt{k_x^2 + k_y^2 + k_z^2}$ .

We match the waves at the point  $(x, y, z)$  to the waves at  $(x, y, 0)$  by decomposing the image at the  $z = 0$  plane into plane waves. This is done by finding the Fourier transform of the image.

$$(2) \quad \tilde{A}_0(k_x, k_y) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \exp[-i(xk_x + yk_y)] A_0(u, v) du dv$$

We expect (and can prove) that we can just extend these plane waves into the  $z$  direction. So we should expect that  $A(x, y, z)$  can be constructed as below.

$$(3) \quad A(x, y, z) = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} \exp[i(xk_x + yk_y + zk_z)] \tilde{A}_0(k_x, k_y) dk_x dk_y$$

it is easy to show that this satisfies the Helmholtz equation  $\nabla^2 A(x, y, z) = -k^2 A(x, y, z)$ . An equation we have all come to know and love. Notice that if  $z \mapsto 0$  then  $A \mapsto A_0$ , nice. We must require, of coarse, that  $k_z = \sqrt{k^2 - k_x^2 - k_y^2}$ .

But, what do we have here? Equation (3) although rough looking is actually easily carried out on a computer. The operation described in this equation consists of first carrying out a Fourier transform of an image, then multiplying this by  $\exp\left[iz\sqrt{k^2 - k_x^2 - k_y^2}\right]$ , and finally carrying out an inverse Fourier transform on the result. Although clean this non-paraxial approach is usually not what you find people using in their calculations. Instead

of solving the Helmholtz equation they instead approximate it to produce the paraxial equation thusly.

Assume that  $A(x, y, z)$ , which is a signal superimposed upon a plane wave  $e^{izk}$ . So  $A(x, y, z) = \psi(x, y, z)e^{izk}$ . Substitute this into the Helmholtz equation and separate out the  $z$  derivative gives you the following.

$$e^{izk} \nabla_{xy}^2 \psi + e^{izk} \frac{\partial^2 \psi}{\partial z^2} + e^{izk} i2k_z \frac{\partial \psi}{\partial z} + -k_z^2 \psi e^{izk} = -k^2 \psi e^{izk}$$

$$k^2 \cong k_z^2 \quad \text{and} \quad \frac{\partial^2 \psi}{\partial z^2} \ll i2k_z \frac{\partial \psi}{\partial z} \quad \text{then}$$

$$(4) \quad \text{the paraxial equation:} \quad \nabla_{x,y}^2 \psi + 2ik \frac{\partial \psi}{\partial z} \cong 0$$

Now for more about approximations. If we are looking close to the optical axis. (almost always true) we can say that  $k_x$  and  $k_y$  are small compared to  $k$ . So then  $k_z \approx k - k_x^2/2k - k_y^2/2k$ . The approximation is much like saying  $\sin(a) = \tan(a)$  i. And thus:

$$(5) \quad A(x, y, z) = \frac{1}{(2\pi)^2} e^{izk} \iint_{-\infty}^{\infty} \exp[-iz(k_x^2 + k_y^2)/2k] \exp[i(xk_x + yk_y)] \tilde{A}_0(k_x, k_y) dk_x dk_y$$

Notice that in equation (5) if we remove the  $e^{izk}$  we have a solution to the paraxial equation (4). We can throw away the factor of  $\exp[izk]$  which shows up in lots of terms and is just the propagation of the general wave in the  $z$  direction. You can't measure it easily. We have already left out a factor of  $\exp[-i\omega t]$ . I should use  $\psi$ 's in stead of  $A$ 's from now on but, what the hell.

Look at equation (5) we see the form, you have the inverse Fourier transform of a quadratic chirp times the Fourier transform of  $A(u, v)$ . Thinking back to your systems class you recall that the inverse Fourier transform of the product of the Fourier transforms of two functions is the convolution of the two functions. So you look at the chirp and look for its inverse transform. We proceed by completing the square in the exponent a few times.  $\alpha$  and  $\beta$  are just place holders to be replaced when we write the convolution.

$$G(\alpha, \beta) = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} \exp[-iz(k_x^2 + k_y^2)/2k] \exp[i(\alpha k_x + \beta k_y)] dk_x dk_y$$

$$(6) \quad = \frac{1}{(2\pi)^2} \exp \left[ \frac{ik}{2z} (\alpha^2 + \beta^2) \right] \iint_{-\infty}^{\infty} \exp \left[ \frac{-iz}{2k} ((k_x - \alpha k/z)^2 + (k_y - \beta k/z)^2) \right] dk_x dk_y$$

$$= \frac{1}{(2\pi)^2} \frac{i2\pi k}{z} \exp \left[ \frac{ik}{2z} (\alpha^2 + \beta^2) \right]$$

We got such a simple result because the integrals are over infinite range so we can slide the definition of  $k_x$  and  $k_y$  to make the  $\alpha k/z$  and  $\beta k/z$  go away.

So if we go back to equation (5) and write it in terms of the results of the above we have:

$$(7) \quad A(x, y, z) = \frac{i}{\lambda z} \iint_{-\infty}^{\infty} \exp \left[ \frac{i\pi}{\lambda z} ((u-x)^2 + (v-y)^2) \right] A_0(u, v) du dv$$

Now we have the expression defined in all optics texts. It is used to describe the transmission of light from one plane to another. (approximately-but actually pretty good) The reason that I bothered to derive the equation is so that we would have the intermediate results equation (3) and equation (5), which are the forms one wants to use in the computer. (With a little cleaning up.)

Looking closely at this equation we see that if we introduce converging lenses with focal length  $z$  at both the starting and ending planes we will get a Fourier transform.

$$(8) \quad \exp \left[ \frac{-i\pi}{\lambda z} (u^2 + v^2) \right] \quad \text{Is a converging lens because of the "-" sign.}$$

Put the above in front of  $A_0(u, v)$  in equation (7) and  $\exp[-i\pi/\lambda z(x^2 + y^2)]$  in front of equation (7) and you will have the following.

$$(9) \quad A_{ft}(x, y, z) = \frac{i}{\lambda z} \iint_{-\infty}^{\infty} \exp \left[ \frac{-i2\pi}{\lambda z} (xu + yv) \right] A_0(u, v) dudv$$

Which is a Fourier transform of  $A_0(u, v)$ . So we have shown that a lens or focal length  $f$  followed by a gap of length  $f$  and a final lens  $f$  produces a 2 dimensional Fourier transform. The second lens is often left out since we can't usually detect phase.

I have been dragging along the second dimension of these equations even though everything could have been shown without  $v$  and  $y$ . So I will drop them from the discussion to make the equations easier to follow. We will now consider a lens with focal length  $f$  a distance  $f$  from plane one and two, just as in the geometric example. The coordinate on the lens plane will be  $\alpha$ . So using equation (7) a couple times and inserting a lens we have the following.

$$(10) \quad \begin{aligned} A_2(x) &= \frac{i}{\lambda f} \iint_{-\infty}^{\infty} \exp \left[ \frac{i\pi}{\lambda f} (x - \alpha)^2 \right] \exp \left[ \frac{-i\pi}{\lambda f} \alpha^2 \right] \exp \left[ \frac{i\pi}{\lambda f} (\alpha - u)^2 \right] A_1(u) d\alpha du \\ &= \frac{i}{\lambda f} \exp \left[ \frac{i\pi}{\lambda f} x^2 \right] \iint_{-\infty}^{\infty} \exp \left[ \frac{i\pi}{\lambda f} (\alpha^2 - 2\alpha(x + u)) \right] \exp \left[ \frac{i\pi}{\lambda f} u^2 \right] A_1(u) d\alpha du \end{aligned}$$

We carry out the  $\alpha$  integral first by completing the square and shifting the integral again. If you fill in the missing steps you find that the leading  $x^2$  and final  $u^2$  terms are canceled.

$$(11) \quad A_2(x) = \frac{i}{\lambda f} \sqrt{-i\lambda f} \int_{-\infty}^{\infty} \exp \left[ \frac{-i2\pi}{\lambda f} xu \right] A_1(u) du$$

or (Changing back to two dimensions.)

$$(12) \quad A_2(x, y) = \iint_{-\infty}^{\infty} \exp \left[ \frac{-i2\pi}{\lambda f} (xu + yv) \right] A_1(u, v) du \, dv$$

So that's why it works.

### 3. MORE ABOUT FRESNEL

One may justifiably wonder how one goes about using a computer to calculate a Fresnel transform in a reasonable amount of time. The answer is not to use equation (7) which can be time consuming (but very good) but to try to find faster algorithms. Equations (2) and (5) above both are very fast; (5) is fastest so I will use it first. The equation makes use of the somewhat inconvenient “wave number”. We should try to make it have more familiar units like mm, not inverse mm. We need to do this since we are putting an image on the screen of a computer which we can easily convert from pixels to mm. The Fourier transform will appear on the same screen. we would like to express the result in the same units. The last equation (12) shows us a way. We will replace  $k$  with  $2\pi/\lambda$  and  $k_x$  with  $2\pi s/\lambda f$  to give us a variable  $s$  with appropriate units. Now equation (5) looks like the following.

$$(13) \quad A(x, y, z) = \frac{1}{\lambda^2 f^2} \iiint_{-\infty}^{\infty} \exp \left[ -\frac{i\pi z}{\lambda f^2} (s^2 + t^2) \right] \\ \exp \left[ i \frac{2\pi}{\lambda f} (xs + yt) \right] \exp \left[ -i \frac{2\pi}{\lambda f} (us + vt) \right] A_0(u, v) \, ds \, dt \, du \, dv$$

The units look good and now as we do our calculation we can secretly think of using a lens system of focal length  $f$ . Well actually three lenses separated by 4 gaps of size  $f$ ; two lenses with focal length  $f$  separated by a lens of focal length  $f^2/z$ . No one would ever build it for this system to replace a single gap  $z$ , but we can think of it.

We can say that the planes are rasterized with integer pixel coordinates and a pixel pitch  $\Delta$ .

$$(14) \quad A(\Delta j, \Delta k, z) = \frac{\Delta^4}{\lambda^2 f^2} \sum_{l, n, m, p=N/2}^{N/2-1} \exp \left[ -\frac{i\pi \Delta^2 z}{\lambda f^2} (l^2 + n^2) \right] \exp \left[ i \frac{2\pi \Delta^2}{\lambda f} (jl + kn) \right] \\ \times \exp \left[ -i \frac{2\pi \Delta^2}{\lambda f} (lm + np) \right] A_0(\Delta m, \Delta p)$$

All is not just fat boat yet. (hunky dory). In order to do a fast Fourier transform using the Cooley-Tukey algorithm the kernel of the transformation should look like

$$\exp \left[ -i \frac{2\pi}{N} (jl + kn) \right]$$

So what we must do is set  $1/N = \Delta^2/\lambda f$ . Which rather nicely defines  $f$  for any pixel pitch, wave length, and array size. This expression shows up in many contexts. So many, that we sometimes call  $f$  the magic distance and in other more formal settings the canonical distance  $f_c = N\Delta^2/\lambda$ .

$$(15) \quad A(\Delta j, \Delta k, z) = \frac{1}{N^2} \sum_{l,n,m,p=N/2}^{N/2-1} \exp \left[ i \frac{2\pi}{N} (jl + kn) \right] \exp \left[ -\frac{i\pi z}{Nf_c} (l^2 + n^2) \right] \exp \left[ \frac{-i}{N} (lm + np) \right] A_0(\Delta m, \Delta p)$$

What you see there is a lens sandwiched between a Fourier transform and an Inverse transform. However there needs to be a little more talk about the lens.

The lens equation (8) above can be put in the form for digital use as the following.

$$(16) \quad \exp \left[ \frac{-i\pi f_c}{Nz} (l^2 + n^2) \right]$$

These clearly differ, one has  $\frac{z}{f_c}$ , and the actual lens expression (16) has  $\frac{f_c}{z}$ . I call the  $z/f_c$  form the dual lens and have incorporated a check box which makes the distinction in the CoherentOptics program, It is used every time I use a Fast Fresnel transform.

In "CoherentOptics" there are three ways to do a Fresnel transform. The best and slowest and most flexible is called "Direct Propagation" Which takes account of the actual size of each pixel. The much faster option is in the "chirp" icon of the script language. (lower right corner) Usually I use the  $\mathcal{F}\mathcal{Z}_{\mathcal{D}}^*\mathcal{F}^{-1}$  set of icons. Where in this case the  $\mathcal{Z}_{\mathcal{D}}^*$  stands for a dual-converging-lens. Which is exactly the situation you see described in equation (15).

One may wonder why one would be so interested in the Fresnel transformation. The answer is that if one were to create the Fresnel transform of some image The complex conjugate of that transform is the hologram of the original image. These can be combined and built up.

In todays usual optics lab the best current devices only display the amplitude. You have the choice of amplitude or phase display. Phase only displays are much more useful. There is more information in the phase. We have published some papers on how to code amplitude information on a phase only device.

Some methods are explained in tutorial #13 of the program.